§12 METAPLECTIC CORRECTION

With all the ingredients –

a symplectic manifold (M,ω)

a prequantum line bundle (L,∇,H) and

a complex polarization $P \subset TM^{\mathbb{C}}$ –

one now can construct the quantum Hilbert space $\mathcal{H}_P$ on which the quantum observables attached to classical observables $F \in \mathcal{E}(M)$ live as selfadjoint operators $q(F)$.

(12.1) Kähler polarizations. Let us first consider the Kähler case. We then we have a Kähler polarization $P \subset TM^{\mathbb{C}}$ on our symplectic manifold (M,ω), that is $P \cap \overline{P} = \{0\}$. In this situation, there exists a unique complex structure on the manifold M (that is the structure of a complex manifold) such that P is the holomorphic polarization, i.e. for the local holomorphic charts

$$\varphi = (z_1, z_2, \dots, z_n) : U \longrightarrow V \subset \mathbb{C}^n, \quad U, V \text{ open}$$

of the complex structure we have

$$P_a = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}, \dots \frac{\partial}{\partial z_n} \right\} \subset T_aM^{\mathbb{C}} = T_aM + iT_aM, \; a \in U.$$

(cf. 11.10.2°)

The complex structure on M can also be described by the family

$$J_a : TM \to TM \ , \ a \in M,$$

where J_a is multiplication with $i \in \mathbb{C}$ coming from the unique complex scale multiplication

$$\mathbb{C} \times T_a M \to T_a M \ , \ a \in M,$$

induced by the holomorphic charts.

In fact, in our case of a Kähler polarization, the pointwise polarizations P_a, $a \in M$, induce an almost complex structure (see last section §11.C, D) $J_a : T_a M \to T_a M$ for each $a \in M$, and define the tensor field $J \in \Gamma(M, \mathrm{End}(TM))$ with $J^2 = -1$. According to a theorem of Newlander - Nirenberg there exists a complex structure on M (given by a holomorphically compatible atlas) inducing J if and only if P is involutive: $[X, Y] \in \mathcal{D}_P(M)$ for all $X, Y \in \mathcal{D}_P(M)$. Since a polarization is involutive we obtain the complex structure on M.

In addition, if we have a prequantum bundle (L, ∇, H) on the complex manifold M, then there exists exactly one complex structure on L

compatible with (L, ∇, H) (cf. (8.6)). Hence, L is a holomorphic line bundle in a natural way. And the polarized sections $s : M \to L$ are nothing else than the holomorphic sections.

The symplectic form ω induces a natural volume ε on M (cf. section 9), and the pre Hilbert space is

$$\{ s \in \Gamma_{hol}(M, L) \mid \int_M \langle s, s \rangle d\varepsilon < \infty \}.$$

It can be completed in order to yield a (separable) quantum Hilbert space $\mathcal{H}_P$. $\mathcal{H}_P$ is a closed subspace of the full Hilbert space of square integrable smooth sections of L considered previously (cf. section 9, in particular page 8).

However, the condition for s to be polarized is not a linear property. It might be (and it happens in general) that fs is no longer polarized even if s is polarized. As a consequence, for a given $F \in \mathcal{E}(M)$ the operator $q(F)$ is not well-defined on (a suitable dense subspace of) $\mathcal{H}_P$. The natural approach to overcome this difficulty is to focus on a certain Lie subalgebra $\mathfrak{O}$ of the Poisson algebra $\mathcal{E}(M)$ of classical observables and to restrict to

$$D = \{ \varphi \in \mathcal{H}_P \mid q(F)\varphi \in \mathcal{H}_P \text{ for all } F \in \mathfrak{O} \}.$$

The previously given $\mathcal{H}_P$ is replaced by the closure of D in $\mathcal{H}_P$. As a result, the quantum Hilbert space can be quite small.

We will not discuss these matters further but rather explain the necessity to make the "metaplectic correction".

(12.2) EXAMPLE: We come back to the the example of $M = T^*\mathbb{R}^n = \mathbb{C}^n$ with the holomorphic coordinates

$$z_j = p_j + i q_j ,$$

cf. (11.10) and the polarization

$$P = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}, .. \frac{\partial}{\partial z_n} \right\} \subset TM^{\mathbb{C}} .$$

We find a potential α for ω,

$$\alpha = \frac{i}{2} \Sigma \bar{z}_j dz_j$$

with the property that $\alpha(X) = 0$ for all polarized $X \in \mathcal{V}_{\bar{P}}(M)$. Our prequantum bundle L is trivial and we can check that

$$s_1(a) = \left(a, \exp\left\{ -\frac{1}{4\hbar} \left(\sum_{j=1}^{n} p_j^2 + q^{j\,2} - 2ip_j q^j \right) \right\}\right) , \quad h > 0,$$

satisfies $\nabla_X s_1 = + \frac{2\pi i}{h} \alpha(X) s_1$ with respect to the connection given by $\frac{1}{h}\alpha$ resp. $\operatorname{Curv}(L, \nabla) = \frac{1}{h}\omega$.

Hence, s_1 is a nowhere zero polarized section. Each section s satisfies $s = f s_1$ and the polarized sections are those which have a holomorphic f $\left(\frac{\partial}{\partial \bar{z}_k} f = 0\right)$. Consequently, the quantum Hilbert space $\mathcal{H}_P$ is

$$\mathcal{H}_P = \left\{ f s_1 \;\middle|\; \int |f|^2 \exp\left(-\frac{|z|^2}{\hbar}\right) d^n z \, d^n \bar{z} < \infty \right\}.$$

Note, that this is a complete space already.

This is the BARGMANN-FOCK REPRESENTATION. It is equivalent to the Schrödinger representation and also to the Heisenberg representation.

The Bargmann-Fock representation is of some use for the quantization of the isotopic oscillator in n dimensions and of a simplified model of the Bose-Einstein field.

(12.3) EXAMPLE: Harmonic Oszillator.
We continue the above example and consider the hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^{n} z_k \bar{z}_k$$

of the harmonic oscillator. Again with $\frac{1}{\hbar}\alpha$ & $\frac{1}{\hbar}\omega$ we obtain $q(H) = \hbar \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$ – the Euler operator – as the quantized H on $\mathcal{H}_P$.

Eigenvalues?

$$\hbar\left(\sum z_k \frac{\partial}{\partial z_k} f\right) = Ef$$

So the eigenvalues are $E_N = N\hbar$ (the homogenous complex polynomials are in the domain of $q(H)$).
This is not quite correct. The observed eigenvalues are $(N + \frac{n}{2})\hbar$ instead. So $\frac{n}{2}\hbar$ is missing (which is the zero point energy).

By comparison we see that as a correct quantized operator $q(H)$ one should take

$$q(H) := \hbar\left(z_j \frac{\partial}{\partial z_j} + \frac{n}{2}\right).$$

This can be achieved by replacing L with the line bundle $L \otimes S$ where $S \to M$ is a geometrically induced complex line bundle over M reflecting the symplectic geometry of (M, ω).

This correction is necessary in other examples, too. It appears quite general in the representation theory of Lie groups. It is called the metaplectic correction.

In order to explain the metaplectic correction in general we need some machinery. The basic concept is the metaplectic structure of a symplectic manifold which is similar to a spin structure of a Riemannian manifold.

Metaplectic Structure:

Let (M,ω) be a symplectic manifold.
A SYMPLECTIC FRAME at $a \in M$ is an ordered basis

$$(u;v) = (u_1, \dots u_n; v_1, \dots v_n) \quad \text{of } T_aM$$

such that $\omega(u_i, v_j) = \delta_{ij}$ & $\omega(u_i, u_j) = \omega(v_k, v_l) = 0$,
$i,j,k,l \leq n$. ("canonical coordinates")

The collection BM_a of symplectic frames at $a \in M$ is in 1-to-1 correspondence to the symplectic group $Sp(n,\mathbb{R})$

$Sp(n,\mathbb{R})$ as a concrete matrix group is nothing else than the group of $2n \times 2n$ real block matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A,B,C,D \ \ n \times n \text{ matrices},$$

satisfying

$$S^T \sigma S = \sigma \quad \text{with} \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ i.e.}$$

$$A^TD - C^TB = 1, \quad A^TC = C^TA, \quad D^TB = B^TC.$$

There is a natural right action of $Sp(n,\mathbb{R})$ on BM_a

$$(u;v) \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto (uA + vC;\ uB + vD)$$

which gives the bijection $Sp(n,\mathbb{R}) \to BM_a$ by fixing a frame $(u^\circ; v^\circ) \in \mathcal{B}M_a$: For each $(u;v) \in \mathcal{B}M_a$ there exists exactly one $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ with $(u^\circ; v^\circ) S = (u; v)$.

The symplectic group is homeomorphic to the product of the unitary group $U(n)$ and a euclidean space. Therefore, the fundamental group is $\mathbb{Z}_2$. Let $Mp(n,\mathbb{R})$ denote the universal covering group of $Sp(n,\mathbb{R})$ which is again a Lie group and let

$$\rho : Mp(n,\mathbb{R}) \longrightarrow Sp(n,\mathbb{R})$$

be the 2-to-1 covering homomorphism. $Mp(n,\mathbb{R})$ is called the METAPLECTIC GROUP.

The collection of all the symplectic frames over a symplectic manifold (M,ω) defines the symplectic frame bundle

$$\mathcal{B}M = \bigcup_{a \in M} \mathcal{B}M_a \longrightarrow M .$$

$\mathcal{B}M$ is a right principal bundle over M with structure group $Sp(n,\mathbb{R})$. This fibre bundle comes automatically with the structure of a symplectic manifold.

(12.4) DEFINITION: A METAPLECTIC STRUCTURE on (M,ω) is a right principal bundle $\widetilde{\mathcal{B}}M$ over M with structure group $Mp(n,\mathbb{R})$ together with a smooth map

$$\tau : \widetilde{\mathcal{B}}M \longrightarrow \mathcal{B}M$$

such that the following diagramm commutes

$$\begin{array}{ccc} \widetilde{\mathcal{B}}M \times Mp(n,\mathbb{R}) & \longrightarrow & \widetilde{\mathcal{B}}M \\ \downarrow \tau\times\varrho & & \downarrow \tau \\ \mathcal{B}M \times Sp(n,\mathbb{R}) & \longrightarrow & \mathcal{B}M \end{array}$$

where the horizontal arrows denote the group actions.

The notion of a metaplectic structure is analogous to the notion of a spin structure of an oriented Riemannian manifold (M,g). Let $\mathcal{B}M_a$ the set of oriented orthonormal bases $(e_1,\dots e_n)$ of T_aM. Then $\mathcal{B}M_a$ is in 1-to-1 correspondence to the special orthogonal group $SO(n,\mathbb{R})$, with a right action of $SO(n,\mathbb{R})$ on $\mathcal{B}M_a$.

The $\mathcal{B}M_a$ fit together to yield the orthonormal frame bundle

$$\mathcal{B}M = \bigcup_{a\in M} \mathcal{B}M_a \longrightarrow M$$

which is a right principal fibre bundle over M with structure group $SO(n,\mathbb{R})$. We know $\pi_1(SO(n,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$.

Now, let $\rho : Spin(n) \longrightarrow SO(n,\mathbb{R})$ the universal covering, a 2-to-1 covering. A spin structure on (M,g) is a right principal fibre bundle $\widetilde{\mathcal{B}M}$ over M with structure group $Spin(n)$ together with a smooth $\tau : \widetilde{\mathcal{B}M} \longrightarrow \mathcal{B}M$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathcal{B}M} \times Spin(n) & \longrightarrow & \widetilde{\mathcal{B}M} \\ \downarrow \tau\times\rho & & \downarrow \tau \\ \mathcal{B}M \times SO(n,\mathbb{R}) & \longrightarrow & \mathcal{B}M \end{array} .$$

In an analogous manner one can define the bundle $\mathcal{L}M$ of Lagrangian frames on M on which $Sp(n,\mathbb{R})$ acts from the left.

If a metaplectic structure $\widetilde{\mathcal{B}}(M)$ has been chosen (in case it it exists at all) it induces a 2-1 cover $\widetilde{\mathcal{L}M} \longrightarrow \mathcal{L}M$ called the bundle of metalinear Lagrangian frames.

For each complex polarization $P \subset TM^{\mathbb{C}}$ on M there is a natural bundle of (complex) frames $\mathcal{R}M$ of the complex, rank n vector bundle $P \to M$

("Reperbündel"), which is a $GL(n,\mathbb{C})$-principal fibre bundle, the bundle of complex linear frames of P.

$\mathcal{R}P$ is a subbundle of $\mathcal{L}(M)$ since all P_a, $a \in M$, are maximally isotropic. $\mathcal{R}P$ is invariant under the action of $GL(n,\mathbb{C})$.

We now use the natural 2-to-1 covering

$$ML(n,\mathbb{C}) \xrightarrow{f} GL(n,\mathbb{C})$$

„inside" $Mp(n,\mathbb{R}) \xrightarrow[\rho]{} Sp(n,\mathbb{R})$. Consequently, the metaplectic structure induces a bundle of metalinear frames $\widetilde{\mathcal{R}P}$ as a subbundle of $\widetilde{\mathcal{L}}M$, and as an $ML(n,\mathbb{C})$-principal fibre bundle.
Again there is the map $\tau : \widetilde{\mathcal{R}P} \to \mathcal{R}P$ such that we have the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{R}P} \times ML(n,\mathbb{C}) & \longrightarrow & \widetilde{\mathcal{R}P} \\ \downarrow \tau \times \rho & & \downarrow \tau \\ \mathcal{R}P \times GL(n,\mathbb{C}) & \longrightarrow & \mathcal{R}P \end{array}.$$

Associated to $\mathcal{R}P$ is the complex line bundle $K := \Lambda^n P$ in which we are interested. (K is associated to the representation

$$\det : GL(n,\mathbb{C}) \longrightarrow GL(\mathbb{C}) = \mathbb{C}^\times.)$$

$\Lambda^n P$ is the n^{th}-exterior product of P, it is called the CANONICAL bundle. A section $s \in \Gamma(M,K)$ can be represented by a function $s^\vee$ on $\mathcal{R}P$ satisfying

$$* \quad s^\vee(w \cdot g) = \det(g^{-1})\, s^\vee(w).$$

for all frames $w = (w_1, \ldots w_n)$ and $g \in GL(n,\mathbb{C})$
And with $\mathcal{E}^{GL}(\mathcal{R}P) = \{ s^\vee \in \mathcal{R}P \mid * \text{ for all } g \,\&\, w \}$ we have a natural isomorphism

$$\mathcal{E}^{GL}(\mathcal{R}P) \longrightarrow \Gamma(M,K),$$

$$s^\vee \longmapsto s, \quad s(a) = s^\vee(w_1, \ldots w_n)\, w_1 \wedge \ldots \wedge w_n .$$

Let $\sqrt[2]{K}$ be a line bundle over M with

$$K = \sqrt[2]{K} \otimes \sqrt[2]{K},$$

it can be described using the square root χ of the character $\det \circ g$ of $ML(n,\mathbb{C})$ with $\chi(1) = 1$.

Finally, the sections in question which one uses as the starting point for the quantum Hilbert space are sections of the line bundle

$$L \otimes \sqrt[2]{K}.$$

So, $\sqrt[2]{K}$ is the bundle S mentioned above.
On $\sqrt[2]{K}$ one defines a "partial" connection respecting only the polarization P, i.e. differentiating in the directions of $\bar{P}$. Together with the connection ∇

on L we get a connection on $L \otimes \sqrt[2]{K}$ and, now, the CORRECTED QUANTUM HILBERT SPACE is the space $\mathcal{H}_P$ of square integrable polarized sections of $L \otimes \sqrt[2]{K}$.

Remarkable: In the case of the oscillator and the Kepler problem the correction

$$L \longmapsto L \otimes \sqrt[2]{K}$$

leads to the right answer.

Existence and uniqueness: The line bundle K defines a (Chern) class $c_1 = [K] \in H^2(M, \mathbb{Z})$ which is the same as the Chern class $c_1 = c_1(M) = [\Lambda^n TM]$, hence independent of the choice of P.

K has a square root $\sqrt[2]{K} = \tilde{L}$ if and only if there is $c \in H^2(M, \mathbb{Z})$ with $2c = c_1$. Hence, the canonical map

$$\pi : H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{Z}_2)$$

induced by $\mathbb{Z} \to \mathbb{Z}_2$ maps c_1 to 0: $\bar{c}_1 := \pi(c_1) = 0$.

In case there is a square root of K, i.e. $\pi(c_1) = 0$, the equivalence classes of such $\tilde{L}$ with $\tilde{L}^2 = K$ are parametrized by by $H^1(M, \mathbb{Z}_2)$.
Hence, if M is simply connected the square root of K is unique up to equivalence.